Bootstrap Confidence Intervals

2025-10-15

Housekeeping

- Coding practice due tonight
- Midterms returned tomorrow!

Bootstrap recap

- Sampling distribution describes how statistic behaves under repeated sampling from population
- Typically cannot repeatedly sample from population, so we obtain **bootstrap distribution** as *approximation* of the sampling distribution of the statistic!

  1. Assume we have a sample $x_1, x_2, \ldots, x_n$ from the population. Call this sample $\boldsymbol{x}$. Note the sample size is n
  2. Choose a large number B. For b in $1, 2, \ldots, B$:

     i. Resample: take a sample of size n with *replacement* from $\boldsymbol{x}$. Call this set of resampled data $\boldsymbol{x}_b^*$

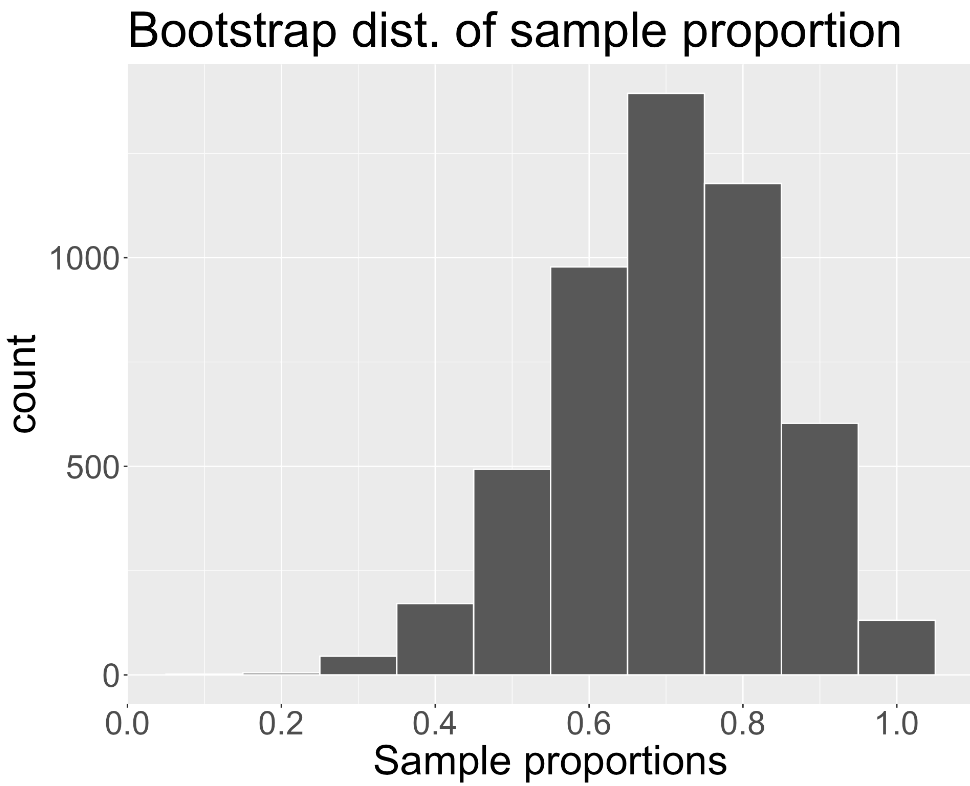
     ii. Calculate: calculate and record the statistic of interest from $\boldsymbol{x}_b^*$

At the end of this procedure, we will have a distribution of **resample or bootstrap statistics**

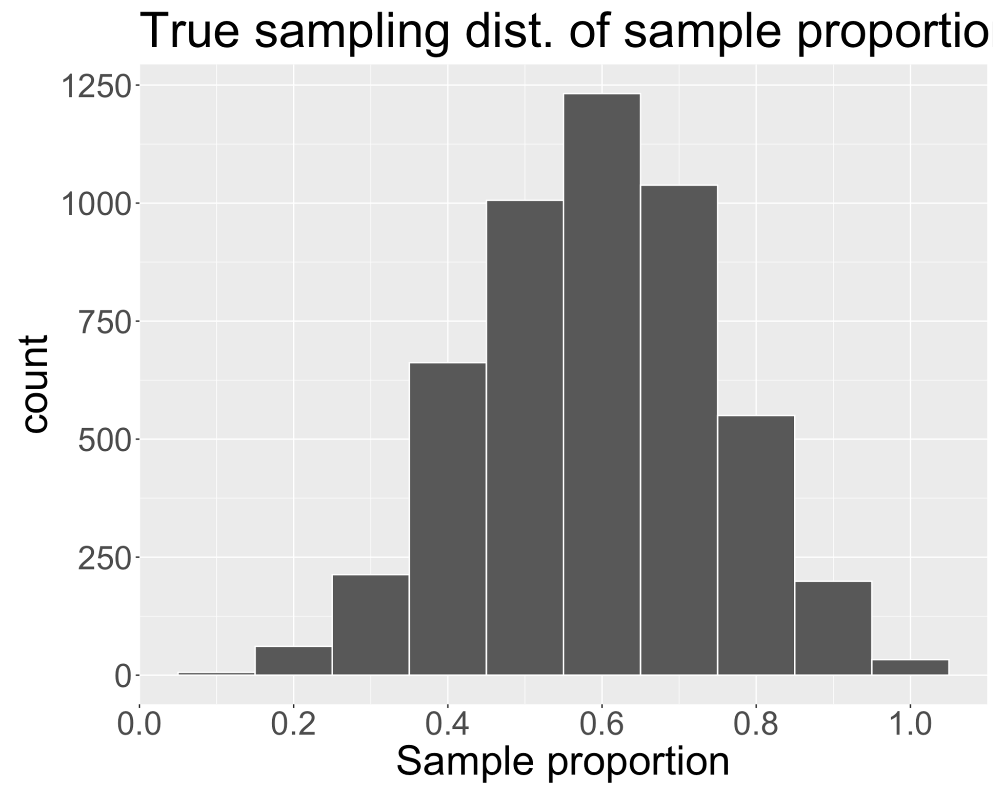
Bootstrap distribution from activity

In our original sample of $n = 10$, we had $\widehat{p_{obs}} = 0.7$.

We have the following bootstrap distribution of sample proportions, obtained from $B = 5000$ iterations:



We can compare to the true sampling distribution (because it's easy for me to re-sample from the population here):



- Notice that our bootstrap distribution isn't a great approximation (maybe $n = 10$ did not yield a representative sample)

Answering estimation question

- Great…but what do we do with the bootstrap distribution?
- Recall our research question: What proportion of STAT 201A pronounced as Middle-“berry”?
  - Could respond using our single point estimate: $\hat{p}_{obs} = 0.7$
  - But due to variability, we recognize that the point estimate will rarely (if ever) equal population parameter
- Rather than report a single number, why not report a range of values?
  - This is **only** possible if we have a sampling distribution to work with!!

Confidence intervals

- Analogy: would you rather go fishing with a single pole or a large net?
  - A range of values gives us a better chance at capturing the true value
- A **confidence interval** provides such a range of plausible values for the parameter (more rigorous definition coming soon)
  - “Interval”: specify a lower bound and an upper bound
  - Confidence intervals are not unique! Depending on the method you use, you might get different intervals

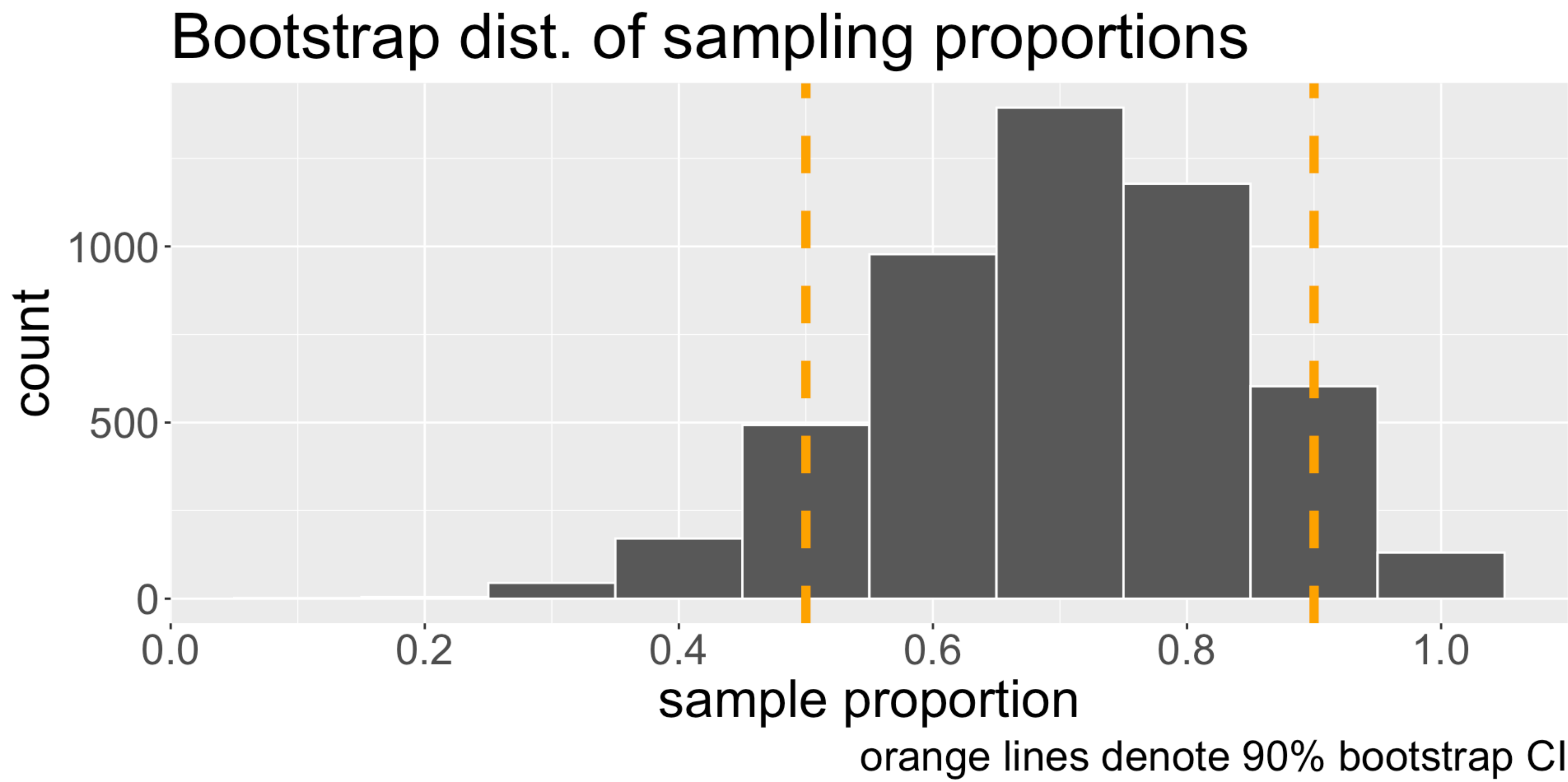
Bootstrap percentile interval

- The $\gamma \times 100\%$ **bootstrap percentile interval** is obtained by finding the bounds of the middle $\gamma \times 100\%$ of the bootstrap distribution
- Called "percentile interval" because the bounds are the $(1 - \gamma)/2 \times 100$ and $(1 + \gamma)/2 \times 100$ percentiles of the bootstrap distribution

- If $\gamma = 0.90$, then the bounds would be at which percentiles?

- For our purposes, "bootstrap confidence interval" will be equivalent to "bootstrap percentile interval"
- `quantile()` function in R gives us easy way to obtain percentiles: `quantile(x, p)` gives us p-th percentile of `x`

Visualizing bootstrap confidence interval



- Our 90% bootstrap CI for p: (0.5, 0.9)

Interpreting a confidence interval

- Our 90% bootstrap CI for p: (0.5, 0.9). Does this mean there is a 90% chance/probability that the true proportion lies in the interval?
  - Answer: NO
- Remember: bootstrap distribution is based on our original sample
  - If we started with a different original sample $\boldsymbol{x}$, then our estimated 90% confidence interval would also be different
- What a confidence interval (CI) represents: if we take many independent repeated samples from this population using the same method and calculate a $\gamma \times 100\ \%$ CI for the parameter in the exact same way, then in theory, $\gamma \times 100\ \%$ of these intervals should capture/contain the parameter
  - γ represents the long-run proportion of CIs that theoretically contain the true parameter
  - However, in real life we never know if any particular interval(s) actually do!

Interpreting a confidence interval (cont.)

- Correct interpretation (generic) of our interval (a, b): We are $\gamma \times 100$ % confident that the population parameter is between a and b.
  - Interpret our bootstrap CI in context
- Again: why is this interpretation **incorrect**? “There is a 90% chance/probability that the true parameter value lies in the interval.”
  - True proportion from census is $p = 0.593$

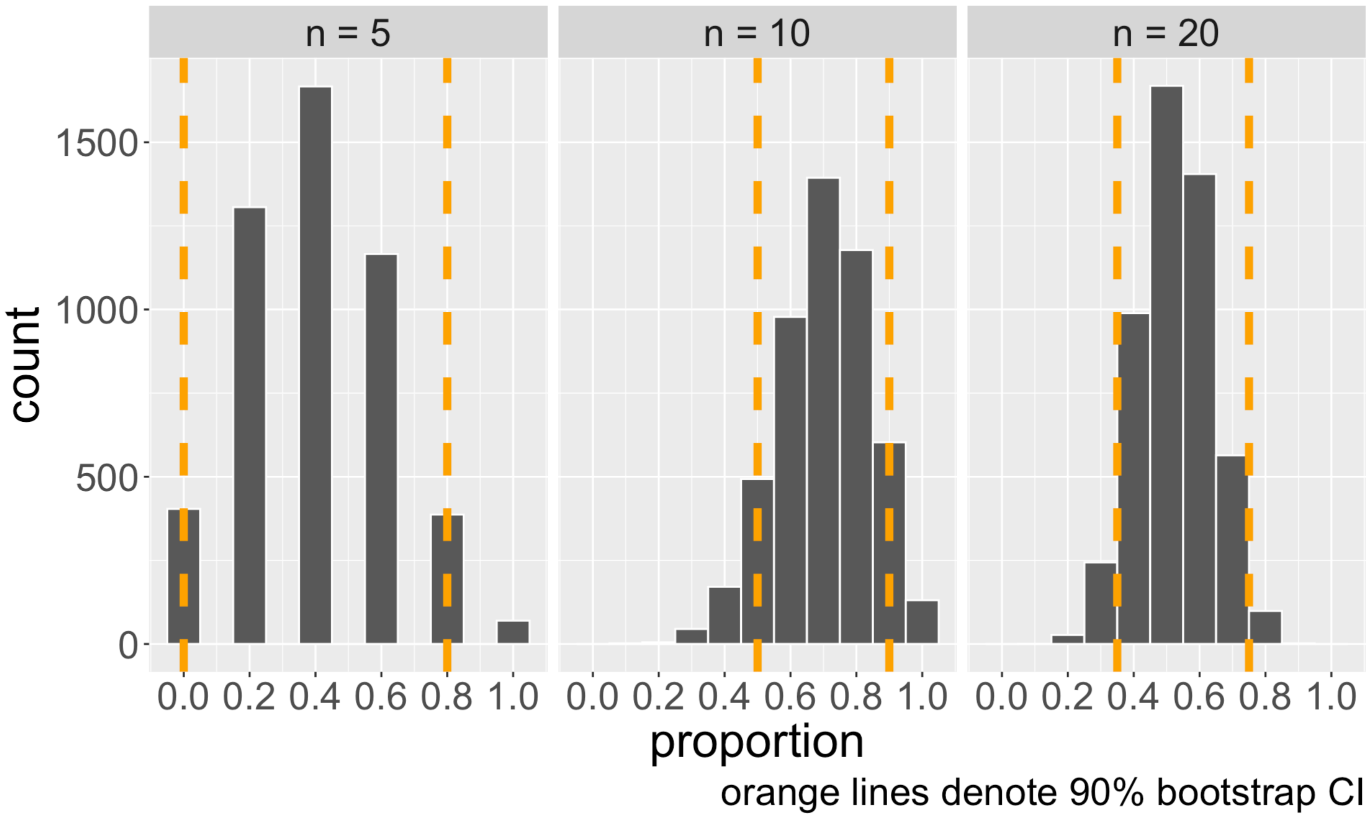
Remarks

- What is a virtue of a "good" confidence interval?
- How do you expect the interval to change as the original sample size n changes?

  How do you expect the interval to change as level of confidence γ changes?
- Once again, a good interval relies on a representative original sample!

Comparing confidence intervals

Comparing changes in 90% bootstrap CI for sample sizes $n = 5, 10,$ and 20.



n	interval
n = 5	(0, 0.8)
n = 10	(0.5, 0.9)
n = 20	(0.35, 0.75)

What do you notice about the bootstrap distributions and CIs as n increases?

Live code + Coding practice!

- Live code:
  - in-line code
  - setting a seed
- You will investigate what happens as we move γ between 0 to 1!

In-line code

In-line code allows us to be reproducible! Suppose I do all this analysis:

```r
x <- c("berry", "burry", "berry", "berry", "berry", "bury", "berry", "burry", "berry", "berry")
n <- length(x)
p_obs <- sum(x == "berry") / n
p_obs
```

```
[1] 0.7
```

I might want to report this value! Rather than “hard-code” the value 0.7 in the white text of my `.qmd`, I want the `.qmd` to update the value dynamically upon rendering!

`.qmd` file

````
```{r}
x <- c("berry", "burry", "berry", "berry", "berry",
       "bury", "berry", "burry", "berry", "berry")
n <- length(x)
p_obs <- sum(x == "berry") / x
```

The observed sample proportion is `r p_obs`.
````

Rendered PDF

```r
x <- c("berry", "burry", "berry", "berry", "berry",
       "bury", "berry", "burry", "berry", "berry")
n <- length(x)
p_obs <- sum(x == "berry") / n
```

The observed sample proportion is 0.7.

- Notice the syntax of inline code! Backticks, r, and the spacing matter!

Setting a seed

When we do random sampling, how can we reproduce our results? By setting a seed!

- "Random" sampling is achieved via a "random number generating function", which takes in a number as input. Kind of like initializing the randomness.

- `set.seed(<integer>)` before doing random sampling will initialize the generator at that specific `integer` input, so subsequent calls to random number generating functions will produce the exact same sequence of "random" numbers