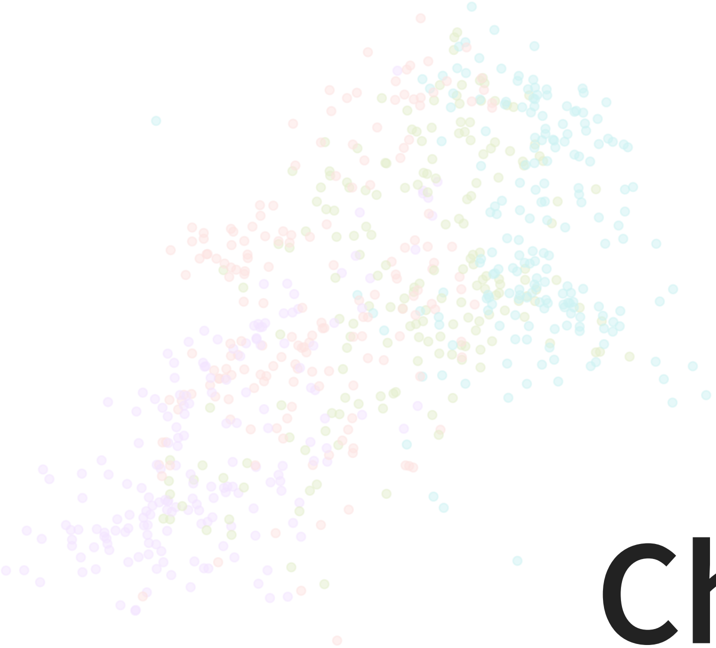
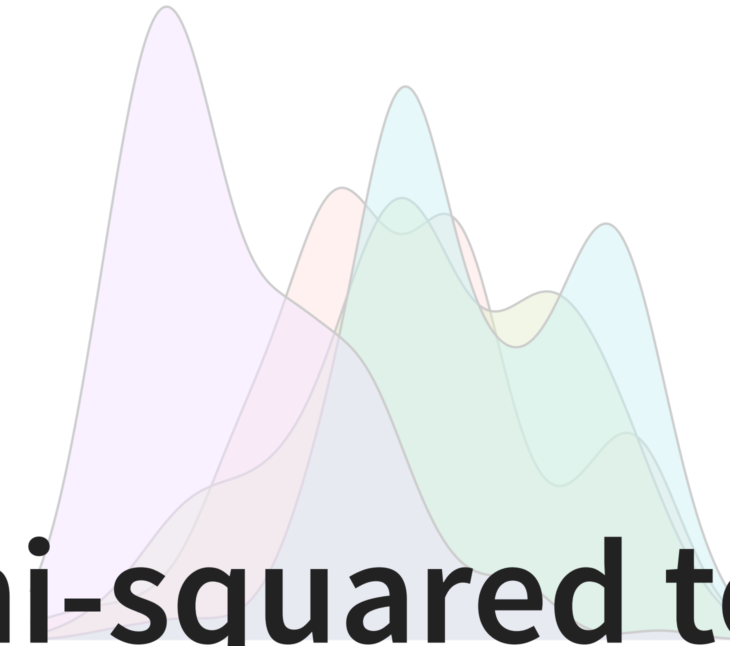
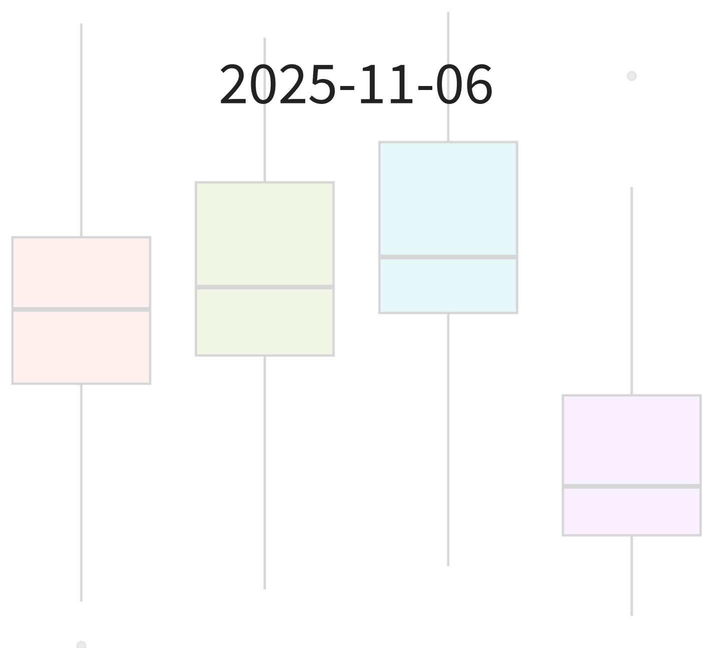
Chi-squared test

2025-11-06

Housekeeping

- Look for project proposal feedback over the weekend!
- Details about Midterm 2 will be announced next week

Chi-squared test for independence

A new test

- Suppose we have two categorical variables:
  - Variable 1 has I levels
  - Variable 2 has J levels
- We are interested in learning if the two variables are associated or not!
  - H_0: the two variables are independent (put into context)
  - H_A: the two variables are associated (put into context)
- Note! We are not interested in a mean/proportion, so CLT will not be helpful here!
- We will build this hypothesis test *backwards.*

Coffee-death data

Recall our data from earlier in the semester:

ORIGINAL RESEARCH Annals of Internal Medicine

Coffee Drinking and Mortality in 10 European Countries

A Multinational Cohort Study

Source: https://www.ncbi.nlm.nih.gov/pmc/articles/PMC5788283/

	Did not die	Died	Total
No coffee	5438	1039	6477
Occasional coffee	29712	4440	34152
Regular coffee	24934	3601	28535
Total	60084	9080	69164

- H_0 : coffee consumption and mortality are independent
- H_A: coffee consumption and mortality are associated
- Note, $I = 3$ rows and $J = 2$ columns

Recall probability

- How might we test these hypotheses?
- Recall from Probability lecture: if events A and B are independent, then $P(A \cap B) = P(A)P(B)$.
    - We will use this idea to obtain "expected counts" in each cell under a world where we assume the two variables are independent.

What is null world?

	Did not die	Died	Total
No coffee			6477
Occasional coffee			34152
Regular coffee			28535
Total	60084	9080	69164

Assuming independence:

$$P(\text{no coffee} \cap \text{die not die}) = P(\text{no coffee}) \times P(\text{die not die}) = \frac{6477}{69164} \times \frac{60084}{69164}$$

- So the **expected *count*** of this cell is the the sample size times this probability:

$$\text{expected count} = 69164 \times \frac{6477}{69164} \times \frac{60084}{69164} = \frac{6477 \times 60084}{69164}$$

Table of expected counts

	Did not die	Died	Total
No coffee	$E_{11} = \frac{6477 \times 60084}{69164} = 5626.69$		6477
Occasional coffee			34152
Regular coffee			28535
Total	60084	9080	69164

- In general, the *expected* counts in cell of row i and column j (denoted E_{ij}) is:

$$E_{ij} = \frac{(\text{row } i \text{ total}) \times (\text{column } j \text{ total})}{\text{Overall total}}$$

- Fill out rest of table!

Table of expected counts

	Did not die	Died	Total
No coffee	$E_{11} = \frac{6477 \times 60084}{69164} = 5626.7$	$E_{12} = 850.3$	6477
Occasional coffee	$E_{21} = 29668.5$	$E_{22} = 4483.5$	34152
Regular coffee	$E_{31} = 24788.9$	$E_{32} = 3746.1$	28535
Total	60084	9080	69164

Towards a test statistic

Remember, a *test statistic* is a quantity that compares our data to null world.

- So let's compare our observed counts to what we would expect under H_0.
- Let O_{ij} represent the observed count in row i and column j
- We might consider the following as a test statistic:

$$\sum_{i,j}(O_{ij} - E_{ij}) \qquad \text{(summing over all } i, j\text{)}$$

- Why might we not like this?

Towards a test statistic (cont.)

- Let's try to fix this issue using the following:

$$\sum_{i,j}(O_{ij} - E_{ij})^2$$

  - This is better, but not perfect because the magnitude of the quantity may be overly influenced by:
    - Highly-represented cells
    - The number of levels I and J

Our test statistic

We will consider the ratio of how far the observed counts are from the expected counts, as compared to the expected count

$$\chi^2 = \sum_{i,j} \frac{(O_{ij} - E_{ij})^2}{E_{ij}}$$

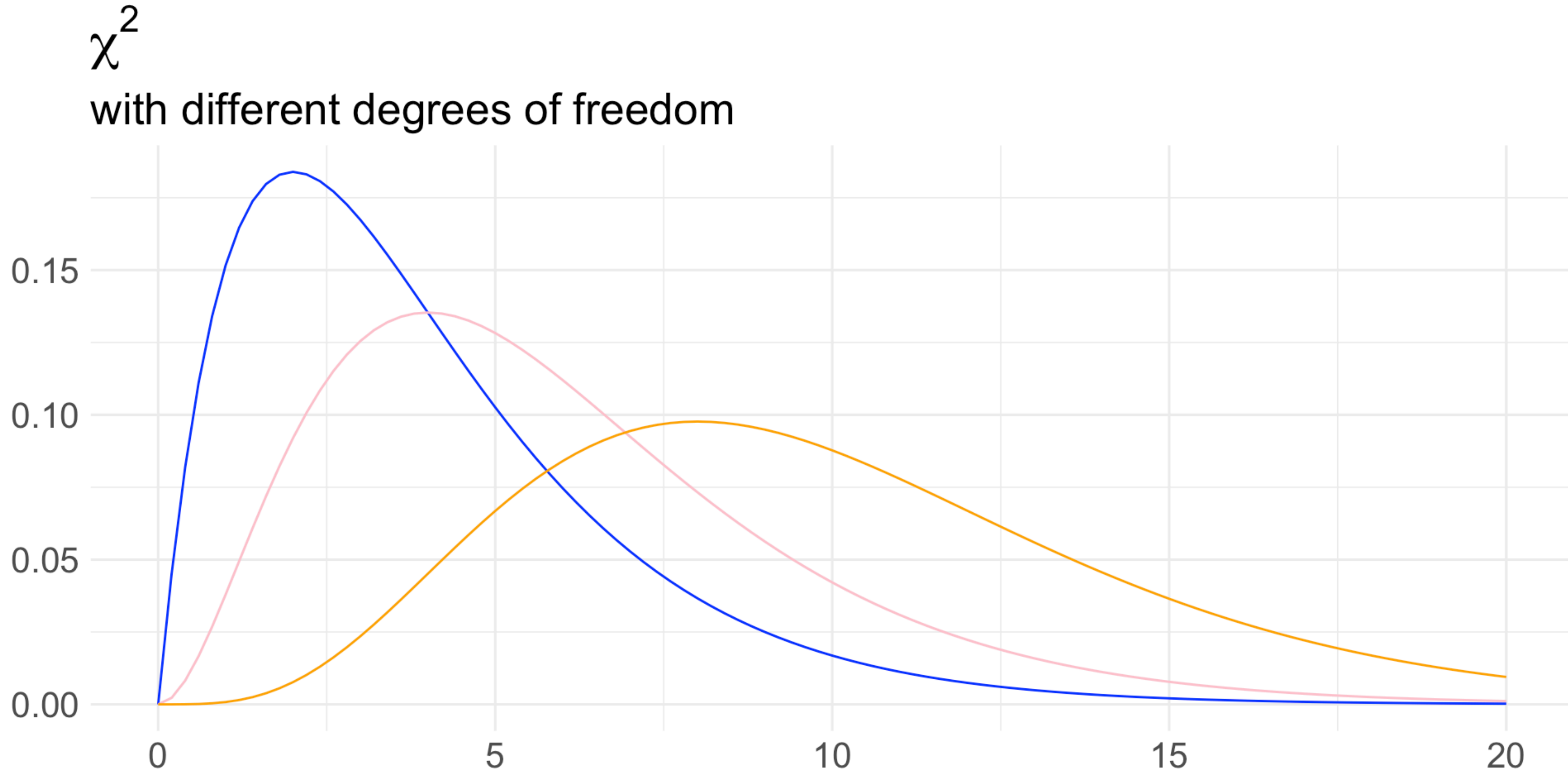
- This is called the **Chi-squared test statistic**
- Let's calculate the value of χ^2 for our data!

$$\begin{aligned} \chi^2 &= \frac{(5438 - 5626.7)^2}{5626.7} + \frac{(1039 - 850.3)^2}{850.3} + \ldots + \frac{(3601 - 3746.1)^2}{3746.1} \\ &= 6.33 + 41.88 + 0.06 + 0.42 + 0.85 + 5.62 \\ &= 55.16 \end{aligned}$$

Understanding the test statistic

- Discuss with someone next to you:
  - What are the bounds of χ^2?
  - Would higher or lower values of χ^2 provide convincing evidence against H_0? Why?
- The **sampling distribution** of this statistic is clearly not Normal. Why?
- If you take STAT 311, you will learn that the null distribution of the χ^2 test statistic is the χ^2_{df} distribution (sorry for terminology)

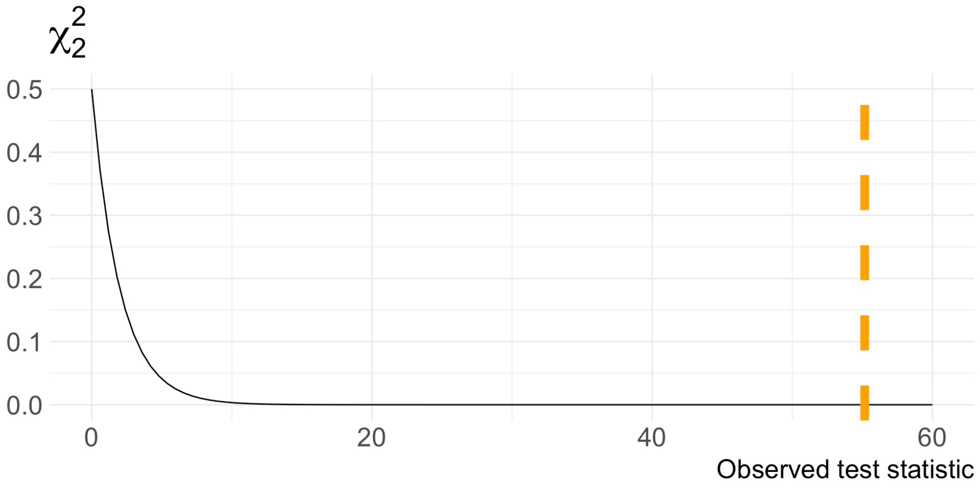
Chi-squared distribution



Null distribution and p-value

When H_0 true, the test statistic $\chi^2 \sim \chi^2_{df}$ where $df = (I - 1) \times (J - 1)$

- That is, $df = (\text{number of rows} - 1) \times (\text{number of columns} - 1)$
- What are the degrees of freedom in our data?
  - $df = (3 - 1) \times (2 - 1) = 2$



- Because the distribution is positive, p-value for this test is always calculated as the probability we are ≥ the observed test statistic
- Use `pchisq(x, df)` function:

```
1 - pchisq(55.15, df = 2)
```

```
[1] 0.000000000001057598
```

Conditions for the chi-squared test

- Even though this isn't a CLT-based result, we are still using a mathematical model (i.e. the χ^2 distribution)
- As a result, we have some conditions that must be met in order to use this test:
    1. Independent *observations* (i.e. random sample)
    2. Large samples: $E_{ij} \geq 5$ for each (i, j) cell
- In our data, the study was a random sample and we definitely had $E_{ij} \geq 5$, so we could indeed perform the Chi-squared test!

Example 2: exploding termites!

- Data come from this study!
- In some termite species, worker termites assume a large share of defense
- This often involves self-sacrifice (i.e. bursting)
- In the species *Neocapritermes taracua*, workers fall into one of two categories based on coloring: “blue workers” and “white workers”
  - The blue coloring comes from a pair of crystal-like structures in the body
  - In defensive mode, both types of workers will burst and emit a toxic fluid
  - Blue termites are thought to be more toxic than white termites

Experiment

- Some scientists wanted to learn about the toxicity of the termites and how they relate to the crystals.
- Designed an experiment where another species of termite (*Labiotermes labralis*) was exposed to a drop of the bursting liquid obtained from one of four sources
  - Observed survival status after 60 minutes

- Four types of burst liquid (treatment):
  1. Blue workers
  2. White workers
  3. Blue workers with crystals removed
  4. White workers with crystals added

- Survival of (*Labiotermes labralis*) (response):
  1. Unharmed
  2. Paralyzed
  3. Dead

Data (cont.)

We have the following data:

	Unharmed	Paralyzed	Dead	Total
Blue workers	3	11	26	40
White workers	31	4	5	40
Blue workers (crystals removed)	26	8	7	41
White workers (crystals added)	17	5	18	40
Total	77	28	56	161

Let's perform a χ^2 test at the 0.05-level to see if bursting liquid source is associated with toxicity!

Conduct test

Define hypotheses

Check conditions for inference.

- H_0: the bursting liquid source and toxicity of the liquid are independent
- H_A: the bursting liquid source and toxicity of the liquid are associated

Conditions

Table of expected counts:

	Unharmed	Paralyzed	Dead	Total
Blue	19.13	6.96	13.91	40
White	19.13	6.96	13.91	40
Blue (w/o crystals)	19.61	7.13	14.26	41
White (w/ crystals)	19.13	6.96	13.91	40
Total	77	28	56	161

Since all expected counts ≥ 5 and it's reasonable to believe independence across termite survival, conditions are met!

Finish test

Obtain (or set-up the calculation) test-statistic, the distribution of the test statistic, and write-code to obtain p-value.

- Value of test statistic:

$$\chi^2 = \frac{(3-19.13)^2}{19.13} + \frac{(11-6.96)^2}{6.96} + \frac{(26-13.91)^2}{13.91} + \ldots + \frac{(18-13.91)^2}{13.91}$$
$$= 48.66$$

- Distribution of test statistic: χ^2_6
- p-value: $P(\chi^2 \geq 48.66)$

```r
1 - pchisq(48.66, df = 6)
```

```
[1] 0.000000008720311
```

Decision and conclusion!