SLR coefficient estimates

2025-11-12

Housekeeping

- Discuss details of Midterm 2

Recap

- **Linear regression**: statistical method where the relationship between variable x and variable y is modeled as a **line + error:**

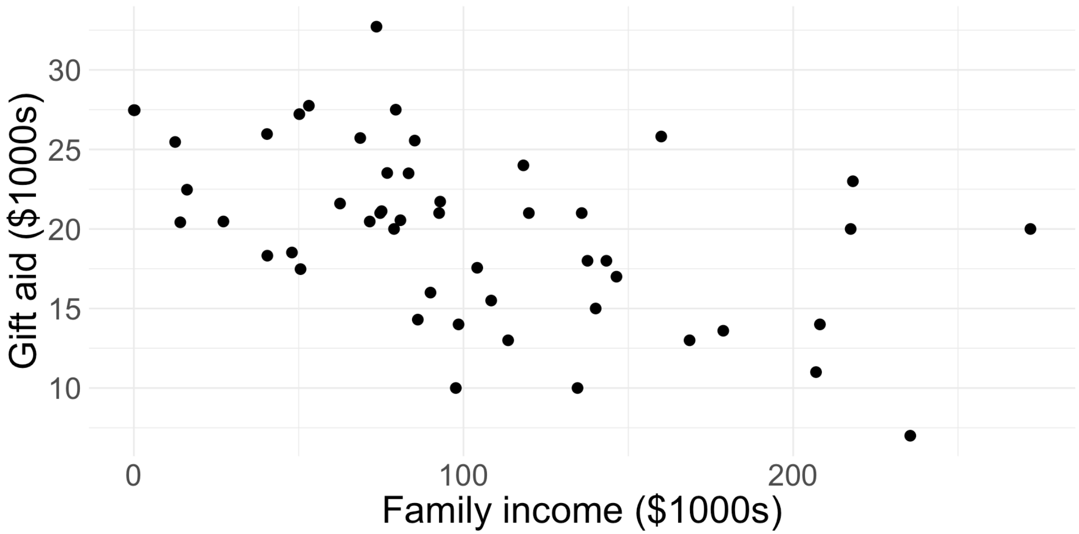
$$y = \beta_0 + \beta_1 x + \epsilon$$

- β_0 and β_1 are population parameters and their corresponding point estimates b_0 and b_1 are estimated from the data
- **Fitted** model: $\hat{y} = b_0 + b_1 x$
- Residual: $e_i = y_i - \hat{y_i}$
- LINE conditions: **L**inearity, **I**ndependence, **N**ormal residuals, **E**qual variance

Example: `elmhurst`

The `elmhurst` dataset from `openintro` provides a random sample of 50 students' gift aid for students at Elmhurst College.

- We will examine the relationship between the family `income` of the student and the `gift aid` that student received (in $1000s)



Write down the linear regression model of interest, in context.

$$\text{gift aid} = \beta_0 + \beta_1 \text{income} + \epsilon$$

Are the first two conditions of LINE satisfied?

Fitting the least-squares line

Parameter estimates

- Like in previous topics, we have to estimate the parameters using data
- We want to estimate β_0 and β_1 using the (x_i, y_i)
  - In practice, we let software do this for us
- However, we *can* derive the least-squares estimates using properties of the least-squares line

Estimating slope and intercept

First obtain b_1:

$$b_1 = \frac{s_y}{s_x} R$$

where:

- s_x and s_y are the sample standard deviations of the explanatory and response variables
- R is the sample correlation between x and y

Then obtain b_0:

$$b_0 = \bar{y} - b_1 \bar{x}$$

where

- $\bar{y}$ is the sample mean of the response variable
- $\bar{x}$ is the sample mean of the explanatory variable

- Take STAT 0211 or 0311 to see where these formulas come from!

Fitting `elmhurst` model (by hand)

Let's obtain this coefficients by hand!

variable	mean	s
family_income	101.78	63.21
gift_aid	19.94	5.46

```
R <- cor(elmhurst$family_income, elmhurst$gift_aid)
R
```

```
[1] -0.4985561
```

What does this value of R tell us?

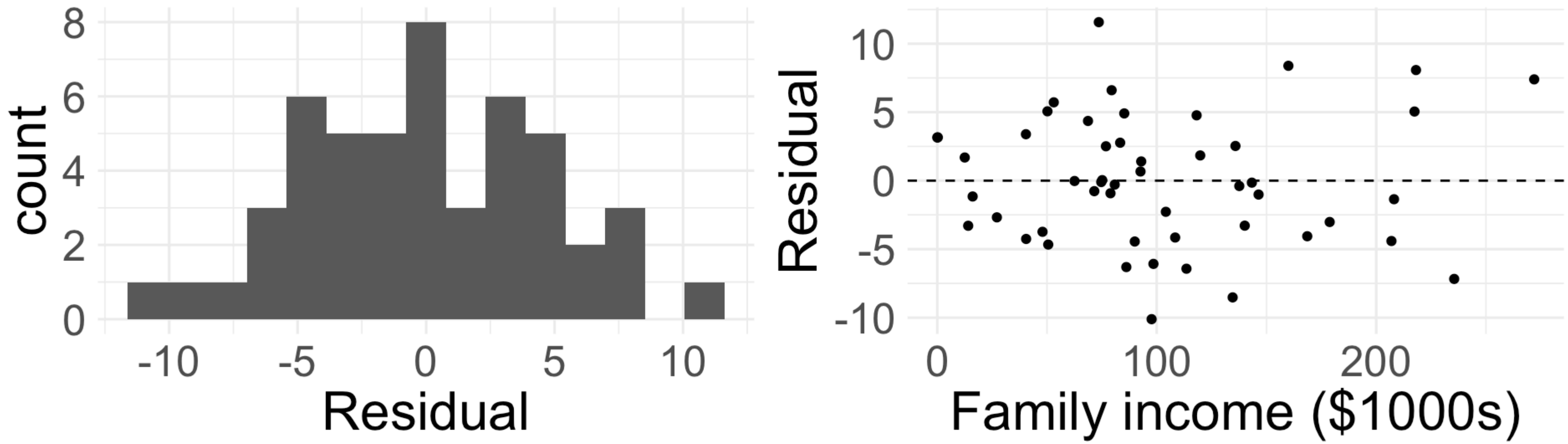
- Set-up the calculations:
  - $b_1 = \frac{s_y}{s_x} R$
  - $b_0 = \bar{y} - b_1 \bar{x}$

- $b_1 = \frac{5.461}{63.206}(-0.499) = -0.043$
- $b_0 = 19.936 - (-0.043)101.779 = 24.319$
- Write out the fitted model!

Example: `elmhurst` model

$$\widehat{\text{gift aid}} = 24.319 - 0.043 \times \text{family\_income}$$

- Before we interpret the coefficients, we should verify that the linear model is appropriate for the data!



Do you believe the last two conditions of LINE are satisfied?

Interpreting parameters

Assuming the SLR model is appropriate, interpreting the parameters (i.e. **coefficients**) is one of *the most* important steps in an analysis!

Intercept interpretation

- To interpret the estimate of the **intercept** b_0, simply plug in $x = 0$:

$$\begin{aligned} \hat{y} &= b_0 + b_1 x \\ &= b_0 + b_1(0) \\ &= b_0 \end{aligned}$$

- So, the intercept describes the **estimated/expected** value of the response variable y if $x = 0$
  - Be sure to interpret in context!

- Interpret the intercept in our `elmhurst` model
  - For a family with an income of $0, the expected gift aid would be $24.319 (in $1000s), or simply $24319

- The intercept's interpretation only makes sense when a value of $x = 0$ is plausible!
  - This is typically not the case/relevant in many applications (though it is here!)

Slope interpretation

- Let $\hat{y_1}$ be the estimated response for a given value of x, so $\hat{y_1} = b_0 + b_1 x$
- Let $\hat{y_2}$ be the estimated response for $x + 1$:

$$\begin{aligned} \hat{y_2} &= b_0 + b_1(x+1) \\ &= b_0 + b_1 x + b_1 \\ &= \hat{y_1} + b_1 \Rightarrow \\ b_1 &= \hat{y_2} - \hat{y_1} \end{aligned}$$

- Interpretation of estimated **slope** b_1: for a 1 unit increase in the explanatory variable x, we expect the response variable y to change by b_1 units

- Interpret in context the estimated slope coefficient in the `elmhurst` model
  - Fine interpretation: For every \$1000 increase in a family's income, we expect the gift aid (in \$1000) the student receives will *decrease* by about \$0.043
  - Better interpretation: …, we expect that the gift aid the student receives will *decrease* by about \$43

Running SLR in R

We run the model in R, and the output looks something like this:

term	estimate	std.error	statistic	p.value
(Intercept)	24.319	1.291	18.831	0
family_income	-0.043	0.011	-3.985	0

- The estimates b_0 and b_1 are shown in the second column
  - We will use the other columns for HTs and CIs!
- We can also easily add the fitted SLR line to a ggplot:

```
ggplot(elmhurst, aes(x = family_income,
                     y = gift_aid)) +
  geom_point() +
  labs(x = "Family income ($1000s)",
       y = "Gift aid ($1000s)",
       caption = "Least squares line") +
  geom_smooth(method = "lm", se = F)
```

Words of caution

- The estimates from the fitted model will always be imperfect
  - The linear equation is good at capturing trends, no individual outcome will be perfectly predicted
- **Do not try to use the model for x values beyond the range of the observed x!**
  - The true relationship between x and y is almost always much more complex than our simple line
  - We do not know how the relationship behaves outside our limited window

Extrapolation

Suppose we would like to use our fitted model to estimate the expected gift aid for someone whose family income is $1,000,000:

- Find the estimated gift aid (careful with units)
  - $\widehat{\text{gift aid}} = 24.319 + -0.043 \times 1000 = -18.681$
  - This is ridiculous!
- This is an example of **extrapolation**: using the model to estimate values outside the scope of the original data
  - We should *never* extrapolate!

Describing strength the fit

If we fit a model and determine LINE was met, we still need a way to describe how "good" the fit is!

- Recall sample correlation R describes the linear relationship between variables x and y
- We typically use the **coefficient of determination** or R^2 (**R-squared**) to describe *strength* of linear fit of a model
  - Describes amount of variation in y that is explained by predictor x in the least squares line
- It turns out that R^2 in SLR is exactly $\ldots$ R squared (i.e. the square of the sample correlation)
  - What are the possible values of R^2? What are desirable values of R^2?

Example: `elmhurst` model fit

- The sample correlation between `family income` and `aid` is $R =$ -0.499
- So the coefficient of determination is $R^2 = (-0.499)^2 = 0.249$
  - **Interpretation**: using a linear model, about 24.9% of the variability in `gift aid` received by the student is explained by `family income`
- I think this is actually a pretty good model!

Categorical predictor

Thus far, we have assumed that x is numerical. Now let x be categorical.

- Note: we will go out of order and check conditions after fitting and interpreting model

Categorical predictor with two levels

- For now, assume that x is categorical with two levels
- Running example: the `possum` data from `openintro` which has data representing possums in Australia and New Guinea
  - Response variable: `tail_l` (tail length in cm)
  - Explanatory variable: `pop` (either "Vic" for possums from Victoria or "other" for possums from New South Wales or Queensland)
- Maybe we would think to write our regression as

$$\text{tail length} = \beta_0 + \beta_1 \text{pop} + \epsilon$$

- Why doesn't this work?
  - Functions require a numerical input!

Indicator variables

We need a mechanism to convert the categorical levels into numerical form!

- This is achieved through an **indicator variable** which takes the value 1 for one specific level and the value 0 otherwise:

$$\text{pop\_other} = \begin{cases} 0 & \text{if pop} = \text{Vic} \\ 1 & \text{if pop} = \text{other} \end{cases}$$

tail_l	pop	pop_other
38.0	other	1
34.0	Vic	0
36.0	Vic	0
36.5	Vic	0
41.5	other	1

- The level that corresponds to 0 is called the **base level**
  - So `Vic` is the base level

Example: `possum` model

This yields the now "legal" SLR model

$$\text{tail length} = \beta_0 + \beta_1 \text{pop\_other} + \epsilon$$

R will automatically convert categorical variables to indicators! So our estimates are as follows:

term	estimate	std.error	statistic	p.value
(Intercept)	35.935	0.253	142.065	0
popother	1.927	0.339	5.690	0

- Write out the equation of our fitted model

Intercept for categorical x

- Our fitted model is:

$$\widehat{\text{tail length}} = 35.935 + 1.927 \times \text{pop\_other}$$

- Try interpreting the intercept b_0

- What does $\text{pop\_other} = 0$ mean? That the possum is from Victoria!
- So when x is categorical, the interpretation of b_0 is the estimated value of the response variable for the base level of x
- Interpretation: the expected tail length of possums from Victoria is 35.935 cm

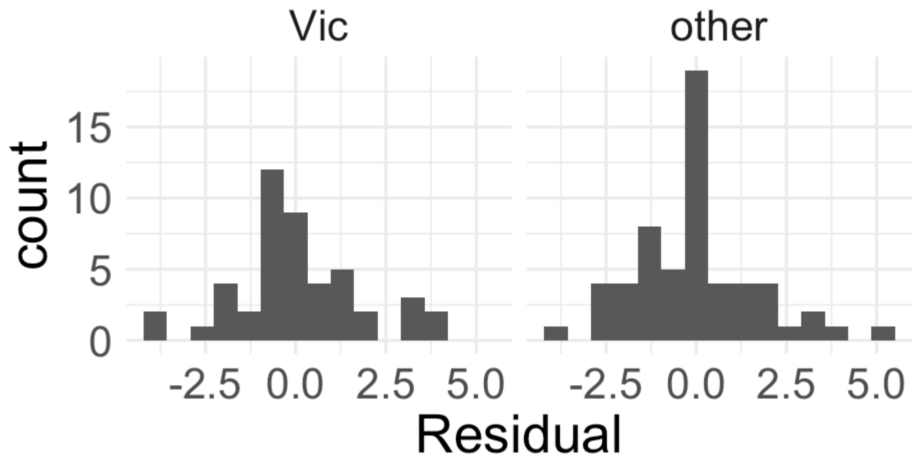
Slope for categorical x

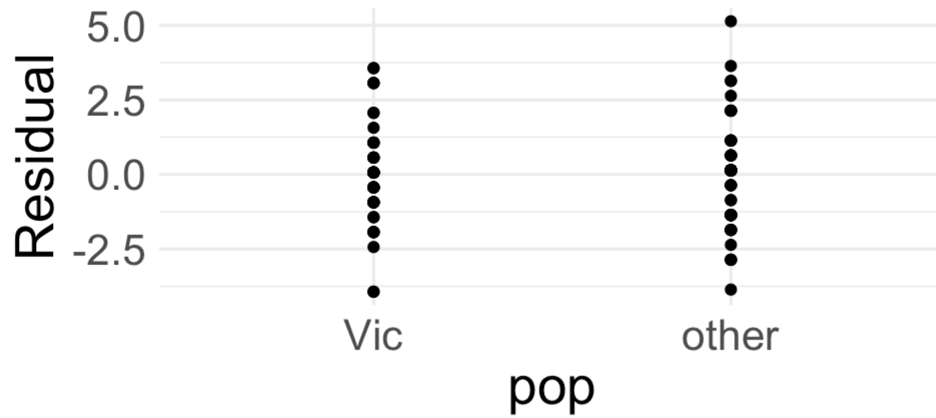
$$\widehat{\text{tail length}} = 35.935 + 1.927 \times \text{pop\_other}$$

- Remember, b_1 is the expected change in y for a one unit increase in x
- What does it mean for pop_other to increase by one unit here?
  - Changing from to a pop value of “Vic” to “other”
- When x is categorical, the interpretation of b_1 is the expected change in y when moving from the base level to that non-base level
- Try interpreting b_1 in context!
  - Interpretation: possums from “other” (i.e. New South Wales/Queensland) are expected to have tail lengths about 1.927 cm longer than possums from Victoria

Assessing linear fit

- When categorical x only has two levels, **L**inearity is always satisfied (yay!)
- **I**ndependence condition is the same before
- We need to evaluate **N**early normal residuals and **E**qual variance for each level
  - Note: the residual plot (right) is uncessary given histogram of residuals





Are all four conditions for SLR met?

Remarks

- When x is categorical, mathematical meaning for b_0 and b_1 are the same as for numerical x, but they have more specific/nuanced interpretations when placed in context
- When x is categorical, SLR is a bit “overkill” (you’ll explore this in homework)
  - As we’ll see next week, the more interest things happen when we have more than one predictor in the model!

Live code